Professional Development Module

Title: Teaching Fractions in Grades 3 - 6

Content and Instructional Shifts: K-5

Targeted Audience: Teachers in grades 3-6

Grade Span: 3-6

Description: Instructor notes; handouts; implementation assignments – based on *Extending Children's Mathematics: Fractions and Decimals* by Empson and Levi

Delivery time: Session 5 of 10 three-hour sessions

The following materials were designed with the intent that the presenter(s) would be educators who have a deep understanding of the mathematical content being addressed at this level.

Session 5 Instructor Notes:

Learning Goals:

- Teachers will understand the content and instructional shifts for teaching fractions resulting from adoption of *Iowa Core Mathematics.*
- Teachers will understand the grade-specific expectations and cross grade-level learning progressions of the *Iowa Core Mathematics* fraction standards.
- Teachers will understand and implement research-based instructional strategies to build students' understanding of fractions and algebra.

Success Criteria:

- Teachers will identify evidence of student understanding of mathematical structures when solving fraction problems.
- Teachers will describe the importance of the properties of operations in *Iowa Core Mathematics.*
- Teachers will plan and implement a fraction lesson with the goal of making students' relational thinking explicit.

Time: 3 hours

Materials:

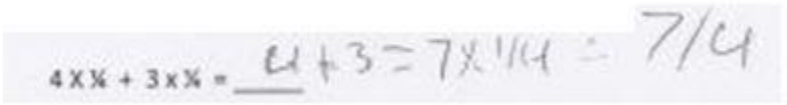
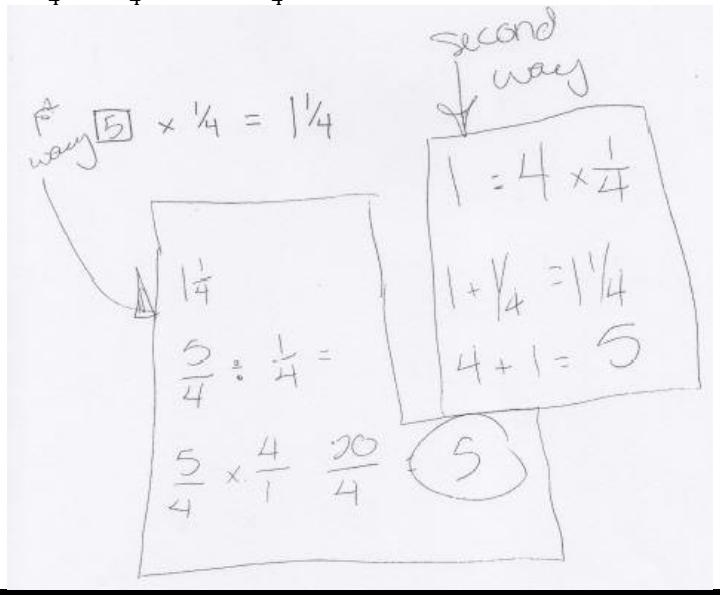
- Book *Extending Children's Mathematics: Fractions and Decimals* by Empson and Levi
- Handout "Iowa Core Mathematics Properties of Operations References" (print in color)
- Handout "Applying the Properties of Operations"
- Handout "Iowa Core Mathematics Content and Practice Shifts Grades K-5"
- Handout "Session 5 Assignment Sheet"
- Instructor Resource "Samples of Student Work for Relational Thinking"
- Student work collected by each participant throughout the course

Session 5 Activity 1
Analyze Student Work from Implementation Assignment 4

Approximate Time: 60 minutes
Key Purpose: To analyze student work and recognize growth in students' understanding of fractions.
Materials:

- Student work collected by each participant

Activity Description	Key Discussion Points
1. Analyze student work from Multiple Groups problem Place participants in groups of three to four teachers. First have each participant share their students' work for a Multiple Groups problem and how they classified it. Pose the following discussion questions: • How did your students do with this problem? Was it easier or more difficult that the last problem you posed (ribbon problem or walking problem)? • Did any of your students use a number line? Did any of your students use a more sophisticated strategy than they have in the past? Give time for each group to share highlights from their discussion with the entire group.	**1. Analyze student work from Multiple Groups problem** The purpose of this activity is for teachers to: • Check their categorization of student work. • Begin to plan next steps based on student understanding. Teachers make instructional decisions to further develop student understanding based on what their students understand as shown in the strategies students use. As before, discussion time is limited and this should be an ongoing discussion throughout the class.
2. Analyze Student Work from Open Number Sentences While in small groups have participants share their students' work for open number sentences and answers to the following questions: • What problem(s) did you present? • What reasoning did your students use to solve the problem(s)? • Did your students show evidence of understanding any of the following mathematical structures and/or properties of operations? Explain. a. $n \times \frac{1}{n} = 1$ and $\frac{n}{n} = 1$ b. $n \times \frac{1}{m} = \frac{n}{m}$ c. $n \times \frac{a}{b} = \frac{(n \times a)}{b}$ d. $\frac{n}{1} = n$ e. Commutative property of operations f. Associative property of operations	**2. Analyze Student Work from Open Number Sentences** As participants work in groups, note examples of evidence of students' understanding of the mathematical structures listed in the left column. Ask select teachers to share their student's work and thinking with the entire class. The following two examples of student work for an open number sentence show evidence of understanding mathematical structures. **Example 1:** This student's use of the equal sign is incorrect as 4 + 3 does not equal $7 \times \frac{1}{4}$, but their work shows evidence of understanding the distributive property of multiplication over addition. $4 \times \frac{1}{4} + 3 \times \frac{1}{4} = (4 + 3) \times \frac{1}{4}$

g. Distributive property of multiplication over addition Give time for each group to share highlights from their discussion with the entire group.	 Example 2: This student solved the problem two ways. The second way shows understanding of $n \times \frac{1}{n} = 1$. It also shows understanding of the distributive property of multiplication over addition. $4 \times \frac{1}{4} + 1 \times \frac{1}{4} = (4 + 1) \times \frac{1}{4}$ 

Session 5 Activity 2
Properties of Operations

Approximate Time: 45 minutes
Key Purpose: To recognize the importance and role of the properties of operations in *Iowa Core Mathematics* at grades K-6.
Materials:

- Book *Extending Children's Mathematics: Fractions and Decimals* by Empson and Levi
- Handout "Iowa Core Mathematics Properties of Operations References"
- Handout "Applying the Properties of Operations"
- Handout "Iowa Core Mathematics Content and Practice Shifts Grades K-5"

Activity Description	Key Discussion Points

Properties of Operations	**Properties of Operations**
Pass out "Iowa Core Mathematics Properties of Operations References" (handout) and have participants note the many times 'properties of operations' or a specific property appear in *Iowa Core Mathematics* standards for grades K-8. Pass out "Applying the Properties of Operations" (handout) and study the examples. The purpose of this page is to show teachers how students might use the properties at different grade levels. Pose the following questions: • Why does *Iowa Core Mathematics* place such an emphasis on the property of operations? • What is the purpose or goal(s) of the following sets of standards? o 1.OA.B.3, 3.OA.B.5, 3.OA.C.7 o 1.NBT.C.4, 1.NBT.C.6, 2.NBT.B.5-7, 3.NBT.A.2-3, 4.NBT.B.5-6, 5.NBT.B.6-7 o 4.NF.B.3 o 4.NF.B.4b, 5.NF.B.4a • When do you think students should learn the formal names for the properties? Tell participants part of their assignment will be to read the shifts for properties of operations, page 3 of "Iowa Core Mathematics Content and Practice Shifts Grades K-5" (handout from session 1).	• Properties of operations are evident throughout the K-12 standards and connect directly to algebra. The same properties apply to whole numbers, decimals, fractions, rational numbers, irrational numbers, and algebraic equations. As a result, developing understanding of the properties of operations in elementary prepares students study algebra in middle school and high school. • In the past, algebraic thinking at grades K-5 often focused on patterns or the function aspect of algebra. Two standards, 4.OA.C.5 and 5.OA.B.3, address patterns and began to prepare students for the study of functions. Most K-5 *Iowa Core Mathematics* 'algebra' standards focus on generalized arithmetic and the properties. • Many content standards for grades K-5 specifically say, "Apply properties of operations..." or "…using strategies based on properties of operations." You will find most of these references in the Operations and Algebraic Thinking Domain, Number and Operations in Base Ten Domain, and the Number and Operations – Fractions Domain. Most of these standards have two goals, 1) developing understanding of the content, and 2) developing understanding of the properties. o **1.OA.B.3, 3.OA.B.5, 3.OA.C.7** The goals are for students to develop automaticity with basic facts **and** develop understanding of the properties of operations. It is developing and using relational thinking to learn basic facts. o **1.NBT.C.4, 1.NBT.C.6, 2.NBT.B.5-7, 3.NBT.A.2-3, 4.NBT.B.5-6, 5.NBT.B.6-7** The goals are for students to develop accurate and efficient ways to compute with multi-digit whole numbers **and** develop understanding of the properties of operations. It is developing and using relational thinking to learn to compute. o **4.NF.B.3** The goals are for students to develop understanding of adding and subtracting mixed numbers **and** develop understanding of the

	properties of operations. For example, $1\frac{2}{5}+2\frac{1}{5} = 1+\frac{2}{5}+2+\frac{1}{5}$. Students can use the associative property to add 1 + 2 and $\frac{2}{5}+\frac{1}{5}$ to get $3\frac{3}{5}$. ○ **4.NF.B.4b, 5.NF.B.4a** These standards do not specifically reference the properties of operations, so they are not included in "Iowa Core Mathematics Properties of Operations References" (handout). However, each standard gives a generalization based on the properties of operations. 4.NF.B.4b states *n × (a/b) = (n × a)/b*. This is an example of the associative property of multiplication as *n × (a x 1/b) = (n × a) x 1/b*. 5.NF.B.4a states *(a/b) × (c/d) = ac/bd*. This is an example of the commutative and associative properties as *(a x 1/b) × (c x 1/d) = (a x c) x (1/b x 1/d).* • First and third grade standards include the following footnote, "Students need not use formal terms for these properties." (*Iowa Core Mathematics* pages 17 and 25.) This may be because first and third grade standards reference a specific property while second and fourth grade standards reference properties of operations in general. Teachers may use the formal names for the properties of operations early, but should not assess students on the names. Fifth grade standards explicitly use names of properties and do not include the footnote referenced above.

Session 5 Activity 3
Making Relational Thinking Explicit

Approximate Time: 70 minutes
Key Purpose: To plan lessons with opportunities to make relational thinking explicit for students.
Materials:

- Instructor Resource "Samples of Student Work for Relational Thinking"
- Student work collected by each participant
- Book *Extending Children's Mathematics: Fractions and Decimals* by Empson and Levi

Activity Description	Key Discussion Points
1. Chapter 4 Discussion Participants read chapter 4 of *Extending Children's Mathematics* (pp. 72-91). Discuss participant's thoughts on the chapter. Pose the	**1. Chapter 4 Discussion** This is a difficult chapter for many teachers. Make sure teachers understand the different student strategies, number sentences, and

following questions: What stands out in this chapter? What is new, interesting, or unique? Is there anything you question or find difficult to understand?

Share the following quote: "Equations that highlight mathematical relationships do not necessarily describe how the problem was solved but rather highlight the relationships that were used in solving the problem" (*Extending Children's Mathematics*, page 111).

connections to the properties of operations. If teachers do not have questions on Chapter 4, have participants explain how Kylie's thinking (pp. 83-84 and 60-61)connects to specific properties of operations.

The following notation is to highlight the complexity of the mathematics in student thinking. This is only here for teachers to see the mathematics behind student strategies; elementary students are not expected to use this complex notation to record their thinking. The next step for teachers is to think about how to help students notate their strategies on a level appropriate to their age. Chapter 5 of *Extending Children's Mathematics* addresses notation for students to some extent.

Problem: Nina had $10\frac{1}{2}$ yards of material and each pillow took $\frac{3}{8}$ yard. How many pillows did she make?

Kylie's thinking step 1: 8 pillows will take 3 yards. $\frac{3}{8}$ x 8 = 3

Mathematical relationships:

$\frac{3}{8}$ x 8 = (3 x $\frac{1}{8}$) x 8 Understand a composite fraction is a multiple of a unit fraction

= 3 x ($\frac{1}{8}$ x 8) Associative property of multiplication

= 3 x 1 Inverse property of multiplication

= 3 Identity property of multiplication

Kylie's thinking step 2: I can get close to $10\frac{1}{2}$ by tripling 3 yards. The number of pillows must be tripled too. $\frac{3}{8}$ x 24 = 9

Mathematical relationships:

$\frac{3}{8}$ x 8 = 3

($\frac{3}{8}$ x 8) x 3 = 3 x 3 Multiplication property of equality

$\frac{3}{8}$ x (8 x 3) = 9 Associative property of multiplication

$\frac{3}{8}$ x 24 = 9 Basic fact

Kylie's thinking step 3: To go from 9 yards to $10\frac{1}{2}$ yards I need $1\frac{1}{2}$ more

	yards, which will make half as many pillows as 3 yards. $\frac{3}{8}$ x 28 = $10\frac{1}{2}$ Mathematical relationships: $\frac{3}{8}$ x 24 = 9 ($\frac{3}{8}$ x 24) + ($\frac{3}{8}$ x 4) = 9 + $1\frac{1}{2}$ Additive property of equality $\frac{3}{8}$ (24 + 4) = $10\frac{1}{2}$ Distributive property of multiplication over addition $\frac{3}{8}$ x 28 = $10\frac{1}{2}$ Basic fact
2. Analyze Student Work Share quote from *Extending Children's Mathematics*, page 90: "Teacher s play a necessary role in making the Relational Thinking in children's strategies explicit by writing equations to represent children's thinking and then questioning students about connections between these equations and their thinking." Show "Samples of Student Work for Relational Thinking" (instructor resource). This resource shows how two different students solved the following problem. Each small cake takes $\frac{3}{4}$ of a cup of frosting. If Betty wants to make 24 small cakes, how much frosting will she need? For each sample of student work ask participants the following questions: • What relationship(s) is evident in the student's work? • How could you as the teacher help students record their strategy to make their relational thinking explicit? Ask participants why teachers should help students use formal notation for recording students' strategies. If time allows have participants look at their own student work to identify strategies using relational thinking. Have them identify two to four examples and answer the following questions: • What relationship is evident in the student's work? • How could you as the teacher help students record their strategy	**2. Analyze Student Work** Student 1: • This student seems to understand 4 cakes take 3 cups of frosting and 24 cakes take 6 times as much frosting or 18 cups. The student states 3 x 8 = 24, but does not seem to use this information to determine the answer. This student seems to have trouble using notation to show the relationship between the number of cups of frosting and number of cakes. • A teacher might help students connect the following notation to this student's work through a class discussion: $\frac{3}{4}$ + $\frac{3}{4}$ = $1\frac{1}{2}$ cups → 2 cakes 2 x $\frac{3}{4}$ = $1\frac{1}{2}$ cups → 2 cakes 4 x $\frac{3}{4}$ = 3 cups → 4 cakes 24 x $\frac{3}{4}$ = 18 cups → 24 cakes 3 fourths + 3 fourths is the same as 2 times 3 fourths. If you double the number of cakes, you need to double the amount of frosting. If you have 6 times the number of cakes, you need 6 times the amount of frosting. Student 2: • This student's work shows evidence of understanding the associative property of multiplication (3 x 7) x $\frac{3}{4}$ = 3 x (7 x $\frac{3}{4}$). The work also shows understanding of the distributive property of multiplication over addition (21 x $\frac{3}{4}$) + (3 x $\frac{3}{4}$) = (21 + 3) x $\frac{3}{4}$. • A teacher might help students connect the following notation to

to make their relational thinking explicit?

this student's work through a class discussion. The teacher may also want to discuss the possibility of finding more convenient groupings such as 4 cakes is 3 cups.

$7 \times \frac{3}{4} = 5\frac{1}{4}$ cups $\rightarrow$ 7 cakes

$21 \times \frac{3}{4} = 15\frac{3}{4}$ cups $\rightarrow$ 21 cakes

$3 \times \frac{3}{4} = 2\frac{1}{4}$ cups $\rightarrow$ 3 cakes

$(21 \times \frac{3}{4}) + (3 \times \frac{3}{4}) = 24 \times \frac{3}{4}$

If you triple the number of cakes, you need to triple the amount of frosting.

21 groups of $\frac{3}{4}$ plus 3 groups of $\frac{3}{4}$ is the same as 24 groups of $\frac{3}{4}$.

Why should teachers help students use formal notation for recording students' strategies?

- Equations help students communicate their thinking to the teacher and other students. (*Extending Children's Mathematics*, pp. 106 & 112)
- Equations help students solve problems. (*Extending Children's Mathematics*, p. 106)
- Equations help students reflect on their work and catch errors. (*Extending Children's Mathematics*, p. 112)
- Equations help students see relationships. They highlight critical mathematics and help students focus on what is essential. (*Extending Children's Mathematics*, pp. 112-113)

2. Plan Next Instructional Steps

Your next implementation assignment is to pose a problem to encourage relational thinking. One problem with several number choices is shown below. Select pairs of numbers you think are appropriate for your students. The pairs of numbers increase in difficulty.

It takes _____ yard of ribbon to make a bow. How many bows can I make with_____ yards?

($\frac{1}{4}$, 12) ($\frac{1}{3}$, 18) ($\frac{1}{3}$, $12\frac{2}{3}$) ($\frac{3}{4}$, 12) ($\frac{2}{3}$, 18) ($\frac{5}{6}$, 35)

3. Plan Next Instructional Steps

Next steps will vary among the teachers depending on the grade they teach and their students' level of understanding. Remind teachers that many students are still depending on drawings. We do not want to push them too fast to abstract notation.

Session 5 Activity 4 Assignment	
Approximate Time: 5 minutes **Materials:** • Handout "Session 5 Assignment Sheet"	
Activity Description	**Key Discussion Points**
1. Reading Assignment: • *Extending Children's Mathematics*, Chapter 5 (pp. 92-113) • "Iowa Core Mathematics Content and Practice Shifts Grades K-5" the shifts under "Properties of Operations" (pp. 3-4) 2. Implementation Assignment 5: • Pose the following problem to encourage relational thinking. Select pairs of numbers you think are appropriate for your students. The pairs of numbers increase in difficulty. It takes ____ yard of ribbon to make a bow. How many bows can I make with____ yards? ($\frac{1}{4}$, 12) ($\frac{1}{3}$, 18) ($\frac{1}{3}$, $12\frac{2}{3}$) ($\frac{3}{4}$, 12) ($\frac{2}{3}$, 18) ($\frac{5}{6}$, 35) • Conduct a class discussion of student work and make students' relational thinking explicit. If students do not use equations, share equations to model specific relational thinking used by students during a class discussion. Discuss how the equation connects to the students' work. • Bring your students' work with you to Session 6 along with a written reflection of what took place in your classroom. Your reflection should include: o the problem you posed, o the notation you or your students used, o a description of the class discussion including connections made between notation and student work, and o a description of what you learned as a teacher. 3. Bring student work from an Equal Sharing problem you have posed that resulted in students getting different equivalent answers (if you have examples).	This assignment is similar to past assignments. The difference is the focus is on making students' relational thinking explicit during the class discussion rather than classifying students' work by strategy after the lesson. Encourage participants to read Chapter 5 before completing the implementation assignment.